Constrained Systems

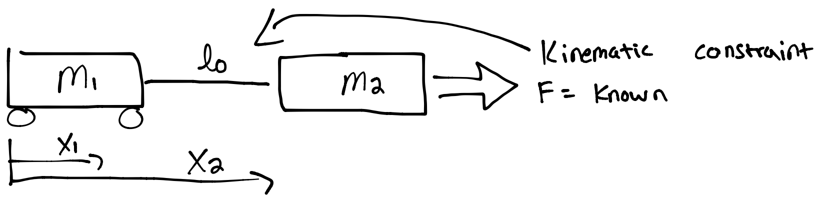
Recall: for each particle $\vec{F}_i = m_i \vec{a}_i$

$$\vec{a}_i = \vec{F}_i / m$$

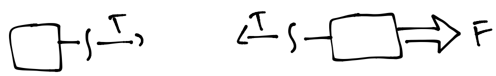
if you Know $\vec{F}_i = \vec{F}_i(\vec{r}_j, \vec{v}_j, t, \text{parameters})$

→ can integrate to get solution (ode45)

What if you have Kinematic constraints?



FBD:



Can't do it: ① $\sum F = m\vec{a}$

$$F - T = m_1 \ddot{x}_1$$

$$\ddot{x}_1 = (F - T)/m_1 \Rightarrow ?$$

1.) Naive approach: LMB : ① $F - T = m_2 \ddot{x}_2$

② $T_1 = m_1 \ddot{x}_1$

Kinematic Constraint $x_2 - x_1 = l_0 \rightarrow \ddot{x}_2 - \ddot{x}_1 = 0$ ③

3 equations for $\ddot{x}_1, \ddot{x}_2, T$ at every instant in time

$$\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}$$

accelerations and constant forces

forces and velocity terms

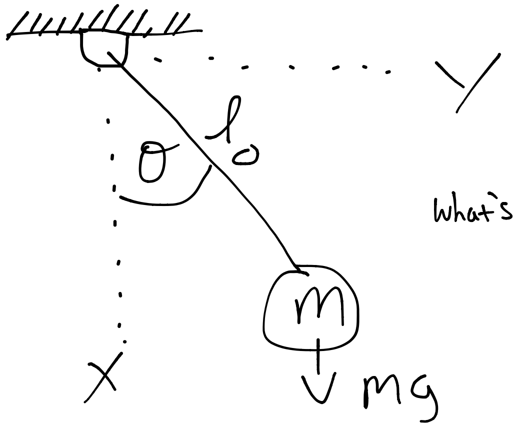
* ode45 in rhs function

<u>Shortcuts</u>: (a) add equations ① and ② to eliminate T

set $\ddot{x} = \ddot{x}_1 = \ddot{x}_2$

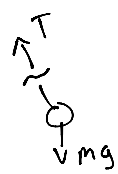
(b) LMB for system: $F = m_1\ddot{x}_1 + m_2\ddot{x}_2$

<u>Example</u>: Simple Pendulum



What's the motion

FBD:



LMB: $\sum \vec{F} = m\vec{a} = mg\hat{\imath} + T(-\hat{e}_r)$

$\hat{e}_r = \dfrac{\vec{r}}{|\vec{r}|} \rightarrow x\hat{\imath} + y\hat{\jmath}$

$\sum \vec{F} = m\vec{a} = m\ddot{\vec{r}}$

$\ddot{\vec{r}} = g\hat{\imath} - \dfrac{T}{m}\dfrac{\vec{r}}{|\vec{r}|}$ $\boxed{\longrightarrow T = ?}$

how do we deal with this

Naive approach: Break $\ddot{\vec{r}} = g\hat{\imath} - \dfrac{T}{m}\dfrac{\vec{r}}{|\vec{r}|}$ into components

$$\ddot{x} = g - \frac{Tx}{m\sqrt{x^2+y^2}} \quad ①$$

$$\ddot{y} = \frac{-Ty}{m\sqrt{x^2+y^2}} \quad ②$$

$$x^2 + y^2 = l_0^2$$

derivative : $2x\dot{x} + 2y\dot{y} = 0$

derivative again : $x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0$ 3

$$\left[\begin{array}{cc|c} m & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & m & \frac{y}{\sqrt{x^2+y^2}} \\ \hline \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{array}\right] \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ T \end{bmatrix} = \begin{bmatrix} mg \\ 0 \\ -\frac{\dot{x}^2+\dot{y}^2}{\sqrt{x^2+y^2}} \end{bmatrix}$$

→ broken into 4 blocks $\begin{bmatrix} m & J \\ J' & 0 \end{bmatrix}$

→ solve for every instant in time (rhs for ode 45)

Shortcuts: 1 generalized coordinate (1 DoF)

LMB: $\vec{F} = m\vec{a}$

$$-T\hat{e}_r + mg\hat{\imath} = m[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta]$$

$r =$ constant → $\dot{r} = 0,\ \ddot{r} = 0$

$$-T\hat{e}_r + mg\hat{\imath} = m(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r)$$

→ dot with $\hat{e}_\theta$

$$\cancel{m}g\underbrace{\hat{\imath}\cdot\hat{e}_\theta}_{-\sin\theta} = \cancel{m}r\ddot{\theta}$$

→ $\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$

Method 2

AMB/o $\qquad \sum \vec{M}_{/o} = \dot{\vec{H}}_{/o}$

$$\vec{r}_{/o} \times \left[-r\hat{e}_r + mg\hat{\imath}\right] = \vec{r} \times \overbrace{\left(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r\right)}^{\vec{a}} m$$

$$\vec{r} \times \hat{e}_r = 0$$

$$-rmg\sin\theta\,\hat{k} = r^2\ddot{\theta}m \xrightarrow{\text{dot with}} \hat{k} \qquad \ddot{\theta} + \frac{g}{r}\sin\theta = 0$$

Method 4

$E_T = \text{constant}$

$$\dot{E}_T = 0 \rightarrow \ddot{\theta} + \frac{g}{r}\sin\theta = 0$$

Method 5

Lagrange Equations